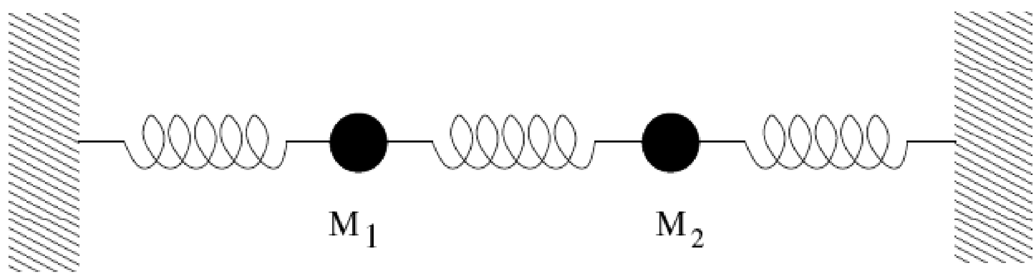
LSU Dept. of Physics and Astronomy
Qualifying Exam
Classical Mechanics Question Bank
(05/2023)

1. A particle is dropped into a hole drilled straight through the center of the earth. Neglecting rotational effects, show that the particle's motion is simple harmonic. Compute the period and give an estimate in minutes. Compare your result with the period of a satellite orbiting near the surface of the earth.

2. Two identical bodies of mass m are attached by identical springs of spring constant k as shown in the figure.



a) Find the frequencies of free oscillation of this system.

b) Mass number 1 is displaced from its position by a small distance a_1 to the right while the mass number 2 is not moved from its position. If the two masses are released with zero velocity, what is the subsequent motion of mass number 2.

3. A planet is in circular motion about a much more massive star. The star under goes an explosion where three percent of its mass is ejected far away, equally in all directions. Find the eccentricity of the new orbit for the planet.

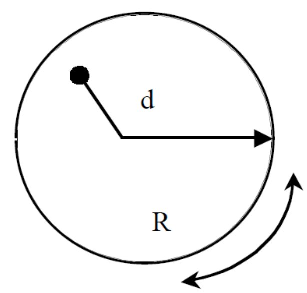
4. A chain of linear density μ (g/cm) is hanging vertically above a table. Its lowest point is at a height h above the table. The chain is released and allowed to fall. Calculate the force exerted on the table by the chain when a length x of chain has fallen onto the surface of the table.

5. An object is dropped from a tower of height h. The tower is located at the equator of the earth. The rotational speed of the earth is Ω.

a) If the acceleration of gravity on the earth ignoring rotation is g, what is the observed acceleration due to gravity on the equator?

b) Even though it is released from rest, this object will not land directly below the point it is dropped. Calculate the amount and direction (N, E, S, W, or elsewhere) of the horizontal deflection of the object. You may assume the deflection is small.

6. Consider a pendulum formed by suspending a uniform disk of radius R at a point a distance d from its center. The disk is free to swing only in the plane of the picture.

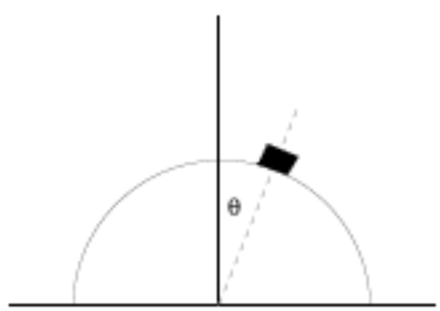


a) Using the parallel axis theorem, or calculating it directly, find the moment of inertia I for the pendulum about an axis a distance d $(0 \leq d < R)$ from the center of the disk.

b) Find the gravitational torque on the pendulum when displaced by an angle ϕ.

c) Find the equation of motion for small oscillations and give the frequency ω. Further find the value of d corresponding to the maximum frequency, for fixed R and m.

7. A heavy particle is placed close to the top of a frictionless vertical hoop and allowed to slide down the loop. Find the angle at which the particle falls off.



8. A solid sphere of radius R and mass M rolls without slipping down a rough inclined plane of angle θ. Take the coefficient of static friction to be μ_S and calculate the linear acceleration of the sphere down the plane, assuming that it rolls without slipping. Calculate the maximum angle, θ_{max}, for which the sphere will not slip.

9. A rocket is filled with fuel and is initially at rest. It starts moving by burning fuel and expelling gases with the velocity u, constant relative to the rocket. Determine the speed of the rocket at the moment when its kinetic energy is largest.

10. A small lead ball of mass m is attached to one end of a vertical spring with the spring constant k. The other end of the spring oscillates up and down with the amplitude A and frequency ω. Determine the motion of the ball after a long period of time. You may assume that the ball is subject to a small amount of damping, he damping force being given by $F = -bv$. Explain the answer.

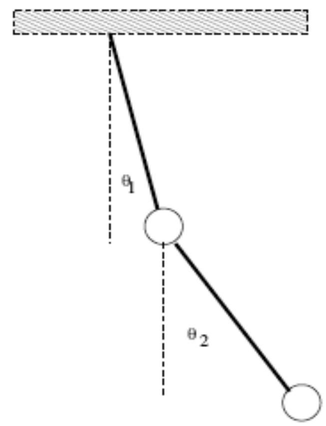
11. A ball of mass m collides with another ball of mass M at rest. The collision is elastic and the motion is one dimensional. The ratio of the masses is a = M/m.

a) Determine how the energy lost by the moving ball depends on the mass ratio a, and find the value of a for which the energy loss is largest. Describe what happens at that value of the mass ratio.

b) Investigate the limiting cases of heavy and light balls and comment on your result.

12. Three points masses of identical mass m are located at (a, 0, 0), (0, a, 2a) and (0, 2a, a). Find the moment of inertia tensor around the origin, the principal moments of inertia, and a set of principal axes.

13. Consider a double pendulum consisting of a mass m suspended on a massless rod of length ℓ, to which is attached by a pivot another identical rod with an identical mass m attached at the end, as shown in the figure. Using the angles θ_1 and θ_2 as generalized coordinates,

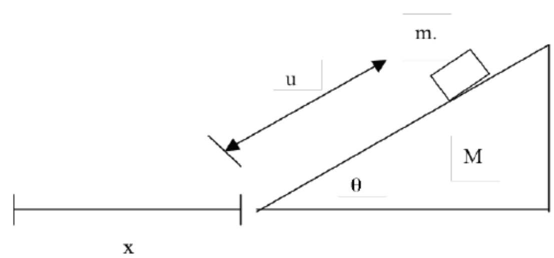


a) Find a Lagrangian for the system

b) Find an approximate Lagrangian that is appropriate for small oscillations and obtain from it the equations of motions when θ_1, $\theta_2 << 1$.

c) Assuming that each angle varies as $\theta_{1,2} = A_{1,2}e^{i\omega t}$, find the frequencies ω for small oscillations.

14. A particle of mass m, at rest initially, slides without friction on a wedge of angle Θ and mass M that can move without friction on a smooth horizontal surface.



a) Find the Lagrangian using the coordinates (x, u) as shown.

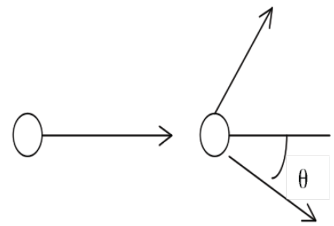
b) Derive the equation of motion from the Lagrangian.

c) What are the constants of motion for the system?

d) Describe the motion of the system.

e) What is the Hamiltonian H (P_x, p_u, x, u)?

15. A target particle of mass m is at rest in the reference frame of the laboratory. It is struck by a projective twice as massive as itself. The scattering is elastic.

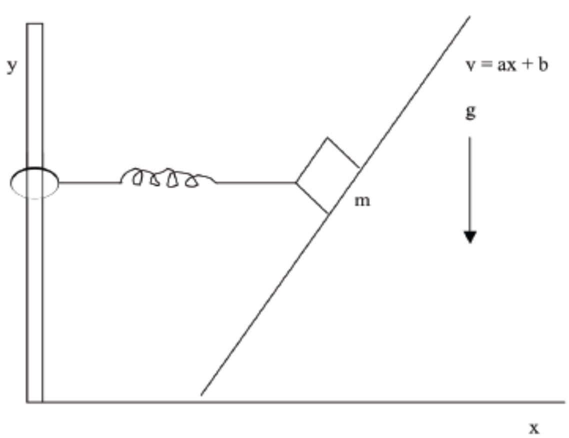


a) What is the largest "scattering angle" θ that the target particle can have after the collision?

b) At what scattering angle does the target particle have the most energy?

c) What is the maximum percentage of its energy that the incident particle can transfer to the scatterer (in lab system)?

16. A mass, *m*, is attached to a spring of spring constant *k* that can slide vertically on a pole without friction, and moves along a frictionless inclined plane as shown in the figure. After an initial displacement along the plane the mass is released. Derive Euler-Lagrangrian equation of motions, and find an expression for the x and y position of the mass as a function of time. The initial displacement of the mass is x_0. You may assume that object never slides down the ramp so far that it strikes the floor.



17. An astronaut is on the surfaces of a spherical asteroid of radius *r* and mean density similar to that of the earth. On the earth this astronaut can jump about a height *h* of 0.5 m. If he jumps on this asteroid, he can permanent leave the surface.

a) Taking the radius of the earth as 6.4×10^6 m, find the largest radius the asteroid can have.

b) How fast could this asteroid rotated and not have the astronaut be flung away from the surface?

18. The relation between the differential scattering cross section in the laboratory and in the center of mass for two non-relativistic particles, m_1 and m_2, with m_2 initially at rest, being scattered by a central force is given by:

$$d\sigma / d\Omega)_L = d\sigma / d\Omega)_C (1+\gamma^2+2\gamma \cos\theta)^{3/2}/(1+\cos\theta),$$

where θ is the scattering angle in the center of mass system. The quantity γ is the ratio of the velocity of the center of mass to the velocity of particle one before the collision measured in the center of mass system.

a) Assume that the collision is inelastic with an energy loss of Q. Under this condition derive an expression for γ in terms of the mass of the two particles, Q and the total energy measured in the laboratory system, E_L.

b) Now assume that the two particles are charged, have equal mass, and the collision is perfectly elastic (Rutherford scattering). Derive the expression for the laboratory differential cross section in terms of the center of mass cross section and the laboratory scattering angle.

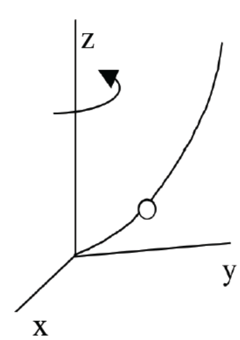
19. A particle of mass m moves in a force field given by a potential energy $V = Kr^s$ where both K and s may be positive or negative.

a) For what values of K and s do stable circular orbits exist?

b) What is the relation between the period P of the orbit and the radius R of the orbit?

20. All parts of this question call for quick and brief answers.

a) A bead of mass m slides along a wire bent into a parabolic shape. The wire is pivoted at the origin and is spinning around the vertical. Identify suitable generalized coordinates and constraints to describe the bead's motion.



b) A system with generalized coordinates q_1, q_2 and q_3 is described by the Lagrangian $L(\dot{q}_1, \dot{q}_2, \dot{q}_3, q_3)$; that is, the Lagrangian is independent of q_1, q_2 and time (explicitly). What are the conserved quantities of this motion?

c) If the kinetic energy is a quadratic function of generalized velocities, $T = \sum_{k\ell} a_{k\ell} \dot{q}_k \dot{q}_\ell$, express $\sum_k \dot{q}_k (\partial T / \partial \dot{q}_k)$ in terms of T.

d) A hollow and a solid sphere, made of different materials so as to have the same masses M and radii R, roll down an inclined plane, starting from rest at the top. Which one will reach the bottom first?

e) Write down the Hamiltonian for a free particle in three-dimensional spherical polar coordinates.

21. All parts of this question call for quick and brief answers.

a) In a bound Kepler orbit, what is the ratio of the average kinetic energy to the average potential energy?

b) Write down the Lagrangian and the Hamiltonian for a free particle of mass m in three-dimensional spherical polar coordinates.

c) A system with generalized coordinates is described by a Lagrangian $L(\dot{q}_1, \dot{q}_2, \dot{q}_3, q_1)$; that is, the Lagrangian is independent of q_2 *and* q_3. What are the conserved quantities of this motion?

d) A hollow and a solid sphere, made of different materials so as to have the same masses and radii, roll down an inclined plane. Which one will reach the bottom first?

e) A satellite initially at a distance R_E (= *radius of the Earth*) above the Earth's surface is moved out to a distance $2R_E$. How does its orbital time period change?

22. A particle of mass m moves under the influence of an attractive force $F = -kr$, where r is the distance from the force center.

a) What are the conserved quantities of this three-dimensional motion?

b) Write down an expression for the total energy.

c) Either from (b) or otherwise, derive an orbit equation relating r to θ.

23. All parts of this question call for quick and brief answers.

a) Positronium is a bound state of an electron and positron. What is the effective mass of positronium?

b) At what point in a bound Kepler orbit is the speed of the satellite at its minimum?

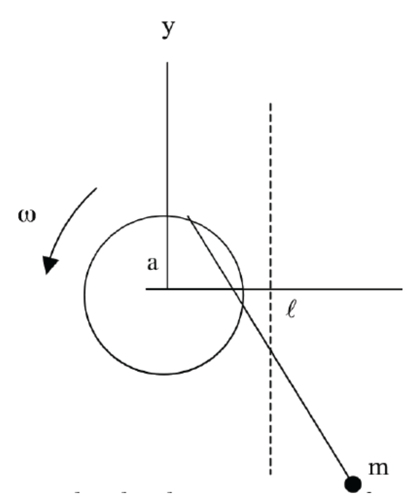
c) If you have two spheres of the same mass and radius, describe a non-destructive test by which you could distinguish between the one that is solid and the one that is hollow.

d) Which way is a particle moving from east to west in the southern hemisphere deflected by the Coriolis force arising from the Earth's rotation?

e) In a bound Kepler orbit, what is the ration of the average kinetic energy to the average potential energy?

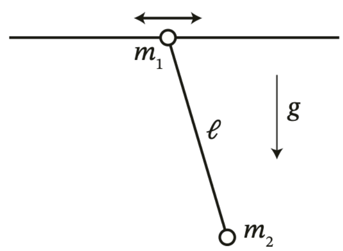
f) Write down the Hamiltonian for a free particle of mass m in three-dimensional cylindrical coordinates.

24. A pendulum, free to swing in a vertical plane, has its point of suspension on the rim of a hoop rotating with constant angular speed ω. Write the Lagrangian and the resulting equations of motion in some suitable coordinates for this system.



25. Obtain the equation of motion for a particle falling vertically under the influence of gravity when frictional forces obtainable from a dissipation function 1/2 kv^2 are present. Integrate the equation to obtain the velocity as a function of time and show that the maximum possible velocity for a fall from rest is $v=mg/k$.

26. Consider the pendulum problem below.

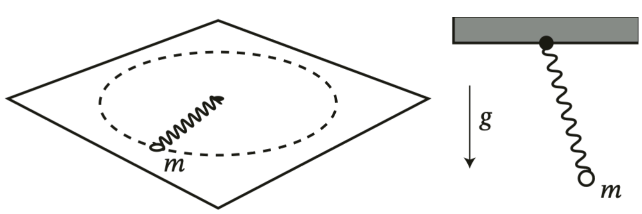


The pendulum (m_2) pictured is subject to gravity and is attached to a slider of mass m_1 which can, without friction, move in the horizontal directions.

a) Write out the constraints, find generalized coordinates, and find the Lagrangian.

b) Assuming small oscillations, derive the equations of motion.

c) Solve the equations of motions derived above normal modes and describe physically what they correspond to and how each mass behaves in general.



(a) (b)

27. A spring of rest length ℓ and spring constant k is connected at one end to a support about which it can rotate and at the other to a mass m.

a) First, consider the geometry confined the surface of a frictionless table. Identify the con- served quantities and write them out in terms of the generalized coordinates. Next, write down and solve the Lagrange equations of motion assuming an initial position $\mathbf{r}_0 = (\ell + x_0, 0)$ and velocity $\mathbf{v}_0 = (0, v_0)$ with $v_0 = 0$. Describe in words what happens if $v_0 \neq 0$. (Assume the mass always stays far from the support.)

b) Next, consider the geometry where the spring is hanging vertically from the support as if it were a pendulum. Identify the conserved quantities and write them out in terms of the generalized coordinates. Next, write down the Lagrange equations of motion and describe the motion about the equilibrium point.

c) Describe in words any similarities or differences in conserved quantities in these two physical scenarios.

28. A bead of mass *m* slides along a smooth wire bent into a parabolic shape $z = \frac{1}{2a} r^2$ (vertical direction z and radial direction r). The wire pivots about the origin and is spinning around the vertical with angular velocity ω.

a) Identify suitable generalized coordinates and constraints to describe the bead's motion. Write down the Lagrangian (but do not solve).

b) Derive the equations of motion, and identify the frequency ω at which the bead, initialized with no radial velocity, remains at a constant radial position for all times.

c) Identify any conserved quantities and compute them.

d) Compute the energy ($\frac{1}{2}mv^2 + mgz$) and the *z*-component of the angular momentum ($m\hat{\mathbf{z}} \cdot \mathbf{v} \times \mathbf{r}$) directly and compare with any conserved quantities in (c).

29. Consider a particle with electric charge q moving in the electrostatic field produced by each of the four charge configurations described below. What components of the particle linear momentum $\mathbf{p} = m\mathbf{v}$, and of the particle angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ will be conserved in each case?

a) An infinite plane of charge, located on the plane $z = 0$.

b) A semi-infinite homogeneous plane $z = 0$ and $y > 0$.

c) An infinite homogeneous solid charged cylinder, with its axis along the y-axis.

d) A finite homogeneous solid charged cylinder, with its axis along the y-axis, and its center at the origin.

e) A homogeneous circular torus, with its axis along the *z*-axis.

30. A particle with internal energy E_{int} is traveling with velocity $\mathbf{V}$ perpendicular to a screen. At a time before it hits the screen, the particle disintegrates losing energy ϵ into kinetic energy, the two components m_1 and m_2 are released isotropically in the center of mass frame (while maintaining momentum conservation). What is the fraction of experimental runs in which particles m_1 hits the screen as a function of $|\mathbf{V}|$? What is the fraction of experimental runs in which they both hit the screen as a function of $|\mathbf{V}|$?

31. All parts of this question call for quick and short answers.

a) If matrices A and B are real symmetric, is the product AB also symmetric? (2 pts)

b) R_{ij} is a second rank tensor. Under a rotation described by the rotation matrix λ_{nm}, express the new components R'_{ij} in terms of the old. (2 pts)

c) Sketch the effective potential for the motion in r of a particle of mass m and angular momentum ℓ in a Yukawa potential $-\frac{k}{r}e^{-\alpha r}$, where k and α are constants. (3 pts)

d) In three dimensions, evaluate $\nabla\times\mathbf{r}$. (3 pts)

32. All parts of this question call for quick and short answers.

a) A particle is projected with momentum mv at a force center so as to pass it at a distance b in the absence of the force. What is the angular momentum of the system? (2 pts)

b) Air rushes away from a high pressure system that forms over Nebraska. Is the resulting circulation clockwise or anti-clockwise? Explain. (2 pts).

c) A billiard ball collides elastically with a stationary one and is deflected by 45 degrees. Sketch and determine the laboratory angle of the second ball's motion after the collision. (3 pts)

d) In three dimensions, evaluate $\nabla \cdot \mathbf{r}$. (3 pts)

33. All parts of this question call for quick and short answers.

a) Write down the Lagrangian and Hamiltonian for a free particle of mass m in three-dimensional cylindrical coordinates. (2 pts)

b) Where in a planet's Keplerian orbit is its speed a maximum? (2 pts)

c) Two spheres have the same diameter and mass, but one is solid and the other hollow. Describe how you would do a non-destructive experiment to distinguish between them. (2 pts)

d) For the coordinate transformation $(x, y)\rightarrow(u, v)$, with $x = uv$, $y = u\sqrt{1-v^2}$, determine the unit vectors $\hat{e}_u$, $\hat{e}_v$ in terms of $\hat{\imath}, \hat{\jmath}$, u, v. (2 pts)

e) A uniform rod of length ℓ stands vertically on a rough floor and then tips over. What is its angular velocity when it hits the floor? Hint: think in terms of energy. (2 pts)

34. A projectile is fired due East from a point on the surface of the Earth at a northern latitude θ with a velocity v_0 inclined at an angle φ to the horizontal. Determine the lateral deflection due to the Coriolis force when the projectile returns to the surface.

35. a) Without explicitly evaluating the individual eigenvalues, find the sum and product of the three eigenvalues of $\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

b) Consider the inertia tensor

$$I = \begin{pmatrix} \frac{1}{2}(A+B) & \frac{1}{2}(A-B) & 0 \\ 3\frac{1}{2}(A-B) & \frac{1}{2}(A+B) & 0 \\ 0 & 0 & C \end{pmatrix}$$

Perform a rotation of the coordinate system by an angle θ about the x_3 axis and evaluate the transformed inertia tensor. Choose θ appropriately to diagonalize the inertia tensor and find the eigenvalues.

36. A bead of mass m is sliding without friction on a vertical loop of radius R. Find the Lagrangian of the bead. Use the method of Lagrange multipliers to find equations of motion and constraint forces.

37. (a) Find the Lagrangian of the spherical pendulum where the length of the pendulum is a constant.

(b) Consider the spherical pendulum as a problem with a Lagrange multiplier. Show that the Lagrange multiplier is the force of constraint in the radial direction. Find the expression of tension in the string.

38. (a) Extremize the following integral to find $y(x)$

$$I = \int_a^b y\sqrt{1+y'^2}\,dx$$

Determine any constants of integration using $y(a) = A$ and $y(b) = B$. Here the prime denotes a derivative with respect to x.

(b) For $f = y(y + 2xy')$, where the prime denotes derivative with respect to x, show that $I = \int_{x_1}^{x_2} f\,dx$ results in same value for all curves joining the end points.

39. Find the equation for the shortest distance between two points on a right circular cylinder. Describe the path and plot it.

40. (a) Given that I is an integral to be extremized

$$I = \int_{x_1}^{x_2} f(y, y', x) dx$$

show that

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y}\right) - \frac{\partial f}{\partial x} = 0.$$

where the prime denotes a derivative with respect to x.

(b) Find the curve $y(x)$ extremizing

$$I = \int_{x_1}^{x_2} (y - yy' + y'^2) dx$$

where the prime denotes a derivative with respect to x.

41. A particle is moving in a potential

$$V(r) = -V_0 e^{-a^2 r^2}$$

where a is a constant. Find the radius of the circular orbit. What is the largest value of angular momentum for which a circular orbit exists? What is the value of effective potential at this critical orbit?

42. A mass m slides down a frictionless plane which is inclined at an angle theta. Show that the normal force from the plane is $mg \cos\theta$.

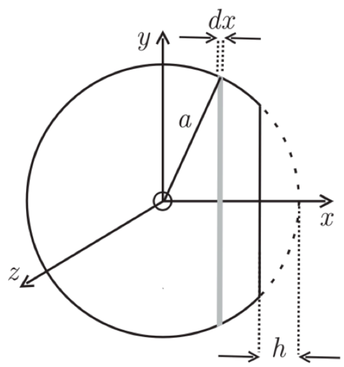
43. A projectile is fired from the origin with velocity $u = (u_1, u_2)$ at time $t = 0$. At the same instant a target is released from rest vertically down from the point (X, Y). Assume that the target falls under a uniform gravitational field and the air resistance can be neglected.

a) Show that the condition for the projectile to hit the target is $\tan\theta = Y/X$ where θ is the angle of projectile launch.

b) Determine the value of initial speed at which the collision occurs (a) at the ground level, and (b) at the apex of projectile's trajectory.

44. A straight section of river flows with speed V between parallel banks a distance D apart. A boat crosses the river, travelling with constant speed u relative to the water, in a direction perpendicular to the current. Determine the boat's path relative to the banks (a) if V is equal to a constant V_0, and (b) if V is zero at the banks and increases quadratically to a maximum V_0 in midstream. In each case determine the distance that the boat is carried downstream in crossing the river, and plot the trajectory.

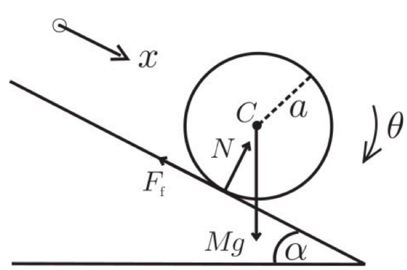
45. A truncated sphere is formed by removing a cap of height h from a uniform sphere of radius a (see figure below) .



(a) Calculate the position of the centre of mass.

(b) Calculate the moment of inertia about the axis of symmetry. Express the results in terms of a and $\epsilon = h/a$.

46. A sphere of mass M, radius a and moment of inertia I is released from rest on an inclined plane at an angle α with coefficient of static friction as μ_s.



a) Suppose the object rolls without slipping. Determine the acceleration of the CM and the required frictional force in terms of M, g, a, α and I. Deduce that there is a critical angle of inclination below which there is pure rolling and above which there is slipping.

b) For $\alpha > \alpha_c$ determine the acceleration of the center of mass and the angular acceleration. Find a relation between the center of mass velocity and the angular velocity.

c) Show that the mechanical energy of the object is constant during rolling but decreases during slipping.

47. A particle in an attractive inverse-square force field $F = -k/x^2$ where k is a positive constant. It is projected along the x-axis with a positive velocity v_o from the point $x_o > 0$. Find the behavior of velocity and determine the equation of trajectory of the particle when

(a) the particle comes to rest at infinity,

(b) the particle comes to rest at a finite value of x and then returns to x_o, and

(c) the particle reaches infinity with a finite speed determined by the system.

Plot behavior of x versus t in each case.

48. Consider a uniform suspension of spherical particles of radius $a = 1.4 \times 10^{-3}$ mm and density $\rho = 4.10 \times 10^3$ kg/m^3 in a lake of depth $D = 2.00$m. The density and viscosity of water are $\rho_m = 1 \times 10^3$ kg/m^3 and $\eta = 1.0 \times 10^{-3}$ Pa s respectively. Assuming that the linear friction applies,

a) Calculate the percentage of particles still in suspension after 12 h.

b) How long does it take for the lake to be clear of suspended particles?

49. A glass aspirator is closed with a stopper through which passes a vertical, precision-made glass tube of uniform cross-section. It is filled with a gas with volume V and bulk modulus K. A closely fitting, smooth, steel ball of mass m is placed in the tube of cross-sectional area A. If the ball is displaced from its equilibrium position and released, it performs underdamped oscillations. Assuming a frictional force linear in velocity, find the period of oscillation in weak damping limit in terms of m, A, V and K.